Your school's crest

Fractions, Decimals and Percentages

Year 7

Booklet 2

Version 0.01.006

Student name: _____

Class: _____

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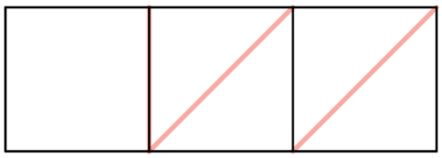
7. Equivalent (equal) fractions – 'scaling up'

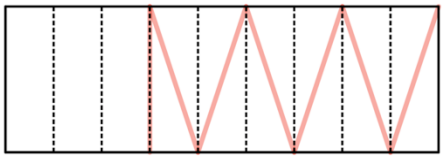
Suppose my delicious block of chocolate is portioned so I can break it into 3 pieces of equal size as shown right.



If I ate *one* piece on Saturday and *one* piece on Sunday, I could say that over the weekend,

$$\text{I ate } \frac{2}{3} \text{ of the block.}$$

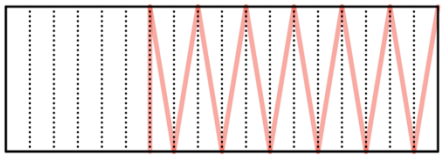
Instead, suppose *the same* block was portioned so I can break it into 9 pieces of equal size as shown right.



If I ate *three* on Saturday and *three* on Sunday, I could say that over the weekend,

$$\text{I ate } \frac{6}{9} \text{ of the block.}$$

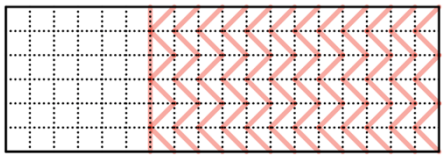
Instead, suppose *the same* block … into 18 pieces of equal size as shown right.



If I ate *six* on Saturday and *six* on Sunday, I could say that over the weekend,

$$\text{I ate } \frac{12}{18} \text{ of the block.}$$

Instead, suppose *the same* block … into 108 pieces of equal size as shown right.



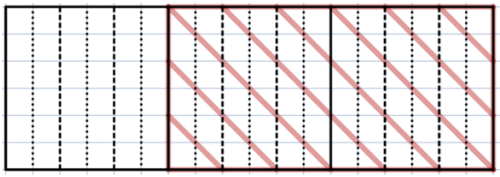
If I ate *thirty-six* on Saturday and *thirty-six* on Sunday, I could say that over the weekend,

$$\text{I ate } \frac{72}{108} \text{ of the block.}$$

In this case we can see that:

$$\frac{2}{3} = \frac{6}{9} = \frac{12}{18} = \frac{72}{108} \text{ (of the block)}$$

Which can be pictured like this:



Multiplying the numerator and denominator of a given fraction *by the same number* will create a new fraction equivalent (equal) to that with which we started.

For example:

$$\frac{3}{4} = \frac{3 \times 3}{4 \times 3} = \frac{9}{12} = \frac{9 \times 5}{12 \times 5} = \frac{45}{60} = \frac{45 \times 9}{60 \times 9} = \frac{405}{540} = \cdots$$

We can think of this as "scaling up" both the numerator and denominator, by the same factor.

Question 1

Draw a diagram that illustrates the fact that:

a) $\frac{2}{5} = \frac{6}{15} = \frac{24}{60}$ b) $\frac{3}{4} = \frac{12}{16} = \frac{24}{32}$ c) $\frac{5}{6} = \frac{25}{30} = \frac{75}{90}$

Question 2

Make *two "different" versions* of the each of the following statements.
Any given fraction can only appear once in a set of three, except for those 'locked in' below. (Do not include any shown earlier in this section.)

a) $\frac{2}{3} = \frac{\quad}{\quad} = \frac{14}{\quad} = \frac{\quad}{\quad}$

b) $\frac{3}{4} = \frac{\quad}{\quad} = \frac{\quad}{32} = \frac{\quad}{\quad}$

c) $\frac{3}{5} = \frac{\quad}{\quad} = \frac{\quad}{45} = \frac{\quad}{\quad}$

d) $\frac{2}{9} = \frac{\quad}{\quad} = \frac{\quad}{\quad} = \frac{\quad}{630}$

e) $\frac{3}{8} = \frac{\quad}{\quad} = \frac{27}{\quad} = \frac{\quad}{\quad}$

f) $\frac{7}{9} = \frac{\quad}{\quad} = \frac{63}{\quad} = \frac{\quad}{\quad}$

Question 3

Each black square is hiding a digit. In each statement, no digit is repeated.
Find as many solutions as you can for each statement.

a) $\frac{\blacksquare}{\blacksquare} = \frac{\blacksquare}{\blacksquare} = \frac{\blacksquare}{\blacksquare\blacksquare}$

b) $\frac{\blacksquare}{\blacksquare} = \frac{\blacksquare}{\blacksquare} = \frac{\blacksquare\blacksquare}{\blacksquare\blacksquare}$

c) $\frac{\blacksquare}{\blacksquare} = \frac{\blacksquare}{\blacksquare\blacksquare} = \frac{\blacksquare}{\blacksquare\blacksquare}$

d) $\frac{\blacksquare}{\blacksquare} = \frac{\blacksquare}{\blacksquare\blacksquare} = \frac{\blacksquare\blacksquare}{\blacksquare\blacksquare}$

e) $\frac{\blacksquare}{\blacksquare} = \frac{\blacksquare}{\blacksquare\blacksquare} = \frac{\blacksquare\blacksquare}{\blacksquare\blacksquare\blacksquare}$

8. Equivalent (equal) fractions – 'scaling down'

We have learned that we can "scale-up" the numerator and denominator of a fraction, by the same factor, and create an equivalent (equal) fraction.

Similarly, we can "scale-down" the numerator and denominator of a fraction, by the same factor, to create an equivalent (equal) fraction.
To do this we *divide* the numerator and denominator by the same number.

For example,

$$\frac{9}{15} = \frac{9 \div 3}{15 \div 3} = \frac{3}{5}$$

Note that both 3 and 5 are prime numbers and so we will not be able to find any more whole numbers that will divide into both the numerator and denominator of $\frac{3}{5}$.

Here is another example,

$$\frac{24}{44} = \frac{24 \div 4}{44 \div 4} = \frac{6}{11}$$

In this case, 6 and 11 are not both prime.
But there is no integer, other than 1, that will divide evenly into 6 and 11.
We say the *greatest common divisor* (GCD) of 6 and 11 is 1.

If a pair (or trio or …) of integers have a greatest common divisor of 1, we say the numbers are *co-prime*.

Having a GCD of 1 is the same as saying the numbers ***do not share any common factors***.

Here is one last example,

$$\frac{12}{70} = \frac{12 \div 2}{70 \div 2} = \frac{6}{35}$$

There is no integer except 1 that divides into 6 and 35, so we say that $\frac{6}{35}$ is in its ***simplest form***.

A fraction is in **simplest form** when the *largest* number that will divide evenly into both the numerator and the denominator is 1.

8.1 Fractions with "easy" numbers as numerators and denominators

To reduce a fraction to its simplest form, one way to proceed is to find a number that will divide into both the numerator and the denominator using your knowledge of the times tables.

To find the simplest form of $\frac{9}{24}$ we note that 3 divides into both 9 and 24, so:

$$\frac{9}{24} = \frac{9 \div 3}{24 \div 3} = \frac{3}{8}$$

Check that the GCD of 3 and 8 is 1, and indeed it is, so $\frac{3}{8}$ is in simplest form.

To find the simplest form of $\frac{18}{24}$ we might first notice that they are both even and so 2 will divide into both 18 and 24, so:

$$\frac{18}{24} = \frac{18 \div 2}{24 \div 2} = \frac{9}{12}$$

Check that the GCD of 9 and 12 is 1. It is not, as 3 will divide into both, so

$$\frac{9}{12} = \frac{9 \div 3}{12 \div 3} = \frac{3}{4}$$

Check that the GCD of 3 and 4 is 1, and indeed it is, so $\frac{3}{4}$ is in simplest form.

(Note that any pair of consecutive numbers will have a GCD = 1.)

Question 1

Write each of the following fractions in their simplest form.

a) $\frac{6}{8}$

b) $\frac{8}{12}$

c) $\frac{12}{42}$

d) $\frac{20}{25}$

e) $\frac{18}{30}$

f) $\frac{28}{63}$

g) $\frac{14}{35}$

h) $\frac{12}{42}$

i) $\frac{15}{45}$

j) $\frac{28}{56}$

k) $\frac{45}{30}$

l) $\frac{80}{16}$

8.2 Fractions to simplest form in "one-step"

To reduce a fraction to its simplest form, find the greatest common divisor (GCD) of the numerator and the denominator and then divide each by that.

We can use the prime factorisation the numerator and the denominator to determine their GCD.

If we needed to find the simplest form of $\frac{42}{90}$:

21 = 2x3x7 and 90 = 2x3x3x5 and so the GCD(42,90) = 2x3

$$\frac{42}{90} = \frac{42 \div 2 \div 3}{90 \div 2 \div 3} = \frac{7}{15}$$

One last example, if we needed to find the simplest form of $\frac{90}{126}$:

90 = 2x3x3x5 and 126 = 2x3x3x7 and so the GCD(90,126) = 2x3x3

$$\frac{90}{126} = \frac{90 \div 2 \div 3 \div 3}{126 \div 2 \div 3 \div 3} = \frac{5}{7}$$

or

$$\frac{90}{126} = \frac{90 \div 9 \div 2}{126 \div 2 \div 9} = \frac{5}{7}$$

Question 1

Calculate the GCD of each pair of integers.

a) 11 and 12

b) 30 and 42

c) 60 and 90

d) 54 and 180

e) 84 and 252

f) 231 and 315

Question 2

Write each of the following fractions in their simplest form.

a) $\frac{54}{180}$

b) $\frac{84}{252}$

c) $\frac{231}{315}$

d) $\frac{231}{252}$

e) $\frac{315}{54}$

f) $\frac{1234}{56789}$

9. Improper fractions - fractions as division

Suppose I said, "Please give me *eight-halves* of a certain type of chocolate bar."

You might, quite rightly, just give me 4 whole bars.

Eight-halves can be written as the fraction $\frac{8}{2}$ and so:

$$\frac{8}{2} = 4$$

Note that $8 \div 2 = 4$.

A fraction, $\frac{a}{b}$, can be interpreted as the division $a \div b$.

So, twelve-fifths, or $\frac{12}{5}$, is the same as $12 \div 5$, so we can write:

$$\frac{12}{5}$$

$$= 12 \div 5$$

$$= 2 \text{ remainder } 2^*$$

$$= 2\frac{2}{5}$$

* the remainder of 2, is $\frac{2}{5}$ (of a whole of 5).

Fractions with a numerator larger in value than the denominator are called *improper fractions*.

Improper fractions can be converted to *mixed numbers* using division, as shown above.

Question 1

Convert each improper fraction to a mixed number.

a) $\frac{17}{2}$ c) $\frac{34}{4}$ e) $\frac{61}{9}$

b) $\frac{28}{3}$ d) $\frac{59}{7}$ f) $\frac{93}{12}$

Question 2

Draw and shade *suitable wholes* that represents:

a) nine fifths b) seven thirds c) $\frac{25}{9}$ d) $\frac{57}{6}$

10. Types of fractions

A *vulgar* (simple/common) fraction is written in the form $\frac{a}{b}$ where:

- the number a is an *integer* and called the *numerator*,
- b is a <u>non-zero</u> *integer* and called the *denominator* and
- the horizontal line is called the *fraction bar* or *division bar*.

For now, we will only discuss positive vulgar fractions.

If $a < b$, for example $\frac{3}{4}$ or $\frac{23}{41}$, the fraction is called a *proper fraction*.

Proper fractions are less than a whole; or numerically, less than 1.

If $a \geq b$, for example $\frac{7}{5}$ or $\frac{32}{11}$ or $\frac{10}{10}$, the fraction is called an *improper fraction*.

Improper fractions are greater than or equal to a whole; or numerically, greater than or equal to 1.

If $a = 1$, for example $\frac{1}{9}$, the fraction is called a *unit fraction*.

Unit fractions are important as you will see later.

Question 1

How many *proper fractions*, in *simplest form*, have a denominator of:

a) 5 b) 10 c) 15 d) 20 e) 100

Question 2

There are different types of fractions other than vulgar (simple/common) fractions; for now, we will call then *mischievous fractions*.

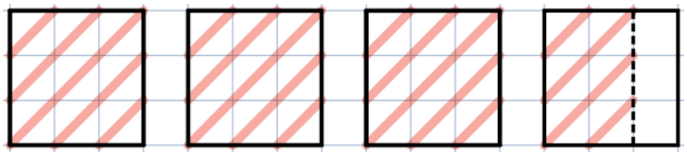
One example of a *mischievous* fraction is $\frac{4\frac{1}{3}}{6}$ (four and one-third sixths) and $\frac{4\frac{1}{3}}{6} = \frac{13}{18}$

Draw and shade *suitable wholes* that illustrates each of the following naughty fractions and then figure out its equivalent proper or improper fraction.

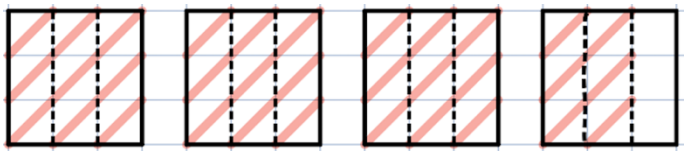
a) $\frac{2\frac{1}{2}}{7}$ b) $\frac{3\frac{1}{4}}{5}$ c) $\frac{2\frac{1}{8}}{3}$ d) $\frac{3\frac{1}{4}}{2}$ e) $\frac{5\frac{1}{5}}{3}$ f) $\frac{1}{2\frac{1}{5}}$ (😳😳)

11. Mixed numbers to improper fractions

The mixed number $3\frac{2}{3}$ can be represented with this diagram:



To convert $3\frac{2}{3}$ to an improper fraction, we can divide the *three wholes* into $\frac{1}{3}$s like this:



So, in total we have *eleven* $\frac{1}{3}$s (3×3+2) which we can write as $\frac{11}{3}$

Note that the each of the wholes is divided equally into the denominator's number of pieces and that gives us a count of how many $\frac{1}{3}$s, in this case we had from the wholes and then we added on the 'extra' *two* $\frac{1}{3}$s.

To convert $5\frac{4}{7}$ to an improper fraction we can imagine:

- divide each of the 5 wholes into seven $\frac{1}{7}$s, so that gives $7 \times 5 = 35$ sevenths
- add on the 'extra' 4 sevenths, $35 + 4 = 39$
- so, $5\frac{4}{7} = \frac{39}{7}$

Question 1

Convert each of the following to improper fractions in simplest form.

a) $3\frac{2}{7}$ b) $5\frac{5}{6}$ c) $7\frac{3}{8}$ d) $8\frac{2}{9}$ e) $4\frac{9}{7}$ f) $3\frac{2\frac{2}{3}}{6}$

12. Fractions are real numbers & exist on the number line

I can just as easily eat $\frac{1}{4}$ of a delicious cake as I can eat 2 cakes.

I can just as easily vertical-jump $\frac{3}{4}$ of a metre as I can 3 metres. (😂)

I can see that $\frac{1}{2}$ one metre is less than $\frac{3}{4}$ of one metre.

Instead of thinking about cake or metres or … - if I just think about the whole as the number 1, then we can happily place fractions on a number line.

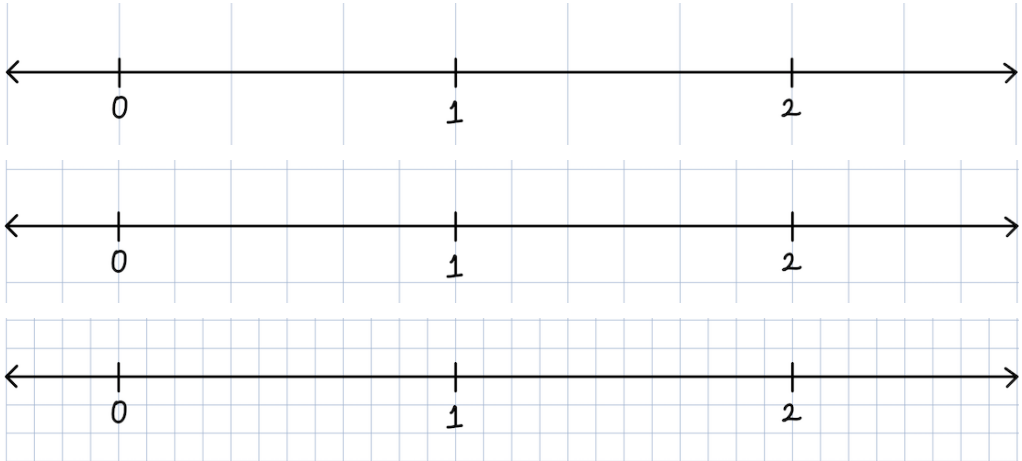
In this section when we state a fraction, like $\frac{6}{7}$ or $\frac{10}{3}$, it will mean $\frac{6}{7}$ of 1, but we will leave of the "off 1".

So, we can think about fractions somewhat like we think about integers, since 5 is *five (lots) of 1*.

Question 1

Mark each of the following fractions on just one of the number lines below.

a) $\frac{3}{4}$ b) $\frac{2}{3}$ c) $\frac{5}{6}$ d) $\frac{7}{12}$ e) $\frac{19}{12}$ f) $\frac{3}{2}$



Question 2

Write these fractions in order of size from smallest to largest.

$$\frac{3}{4}, \frac{3}{2}, \frac{19}{12}, \frac{2}{3}, \frac{7}{12}, \frac{5}{6}$$

Question 3

What is the smallest proper fraction greater than zero?

Question 4

What is the largest proper fraction less than one?

Question 5

How many proper fractions exist that are greater than 0 and less than 1? Provide an explanation for your answer to help another person understand why your answer is correct.

Question 6

Do you think there are more, less or the same number of proper fractions between 0 and $\frac{1}{4}$ compared to between 0 and $\frac{3}{4}$? Why?

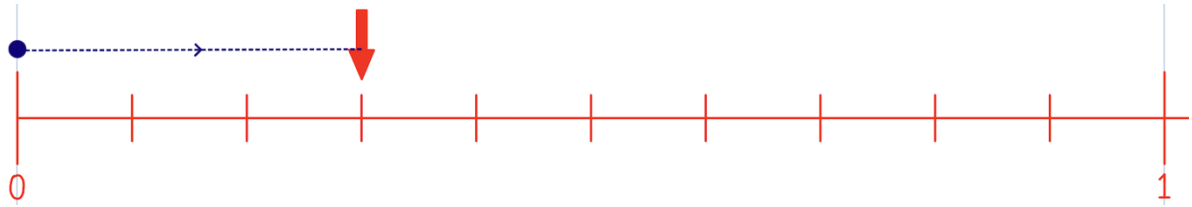
13. The number line as a ruler

I imagine you have realised long ago that part of the number line doubles as a tool for measurement, aka a ruler or tape measure.

On a ruler, the 1 is not just the *general* 1 'unit', that can be of any size you choose, but rather 1 specific unit – like a metre or inch or …

Shown below is *start* of a long ruler that measures *distance* in the very rarely used unit called the *pinocchio* (shortened to *pch* 🤥).

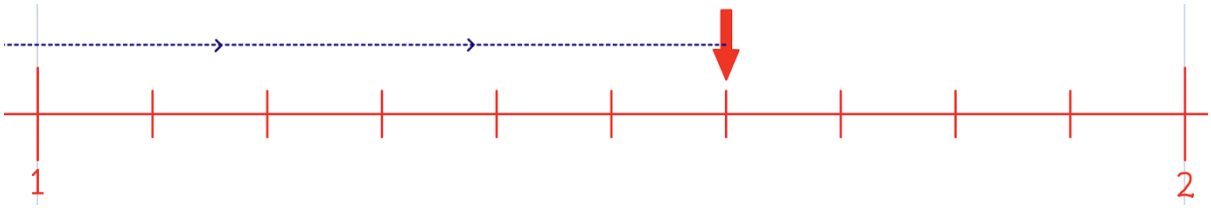
The red arrow has moved from the start (0) and stopped where you see it.



The red arrow has moved

$$\frac{3}{10} \text{ pch.}$$

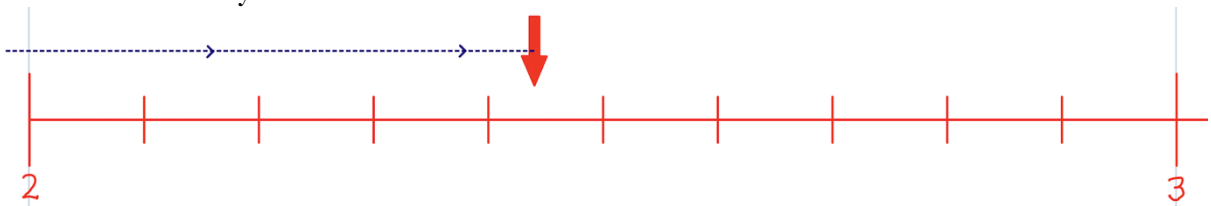
Shown below is a section of the ruler from 1 pch to 2 pch.
The red arrow started again and stopped where you see immediately it.



It has now moved a total of

$$1 + \frac{6}{10} \text{ pch.}$$

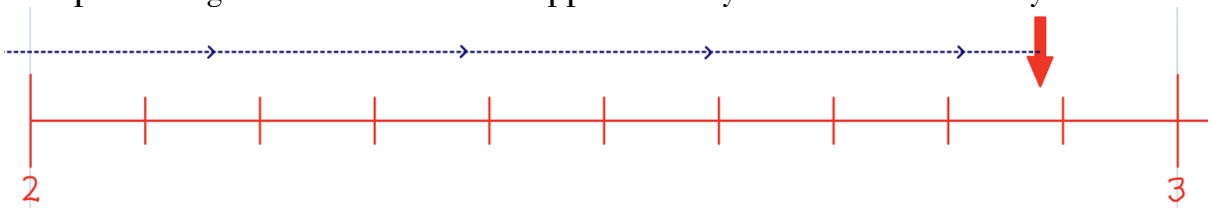
The arrow kept moving forwards, past 2 on the ruler and stopped again where it is seen immediately below.



It is difficult to be accurate, so the best we can say is the red arrow has now moved

$$\textit{more than } 2 + \frac{4}{10} \text{ pch and } \textit{less than } 2 + \frac{5}{10} \text{ pch.}$$

There is no stopping the red arrow!
It kept moving forwards and then stopped where you see it immediately below.



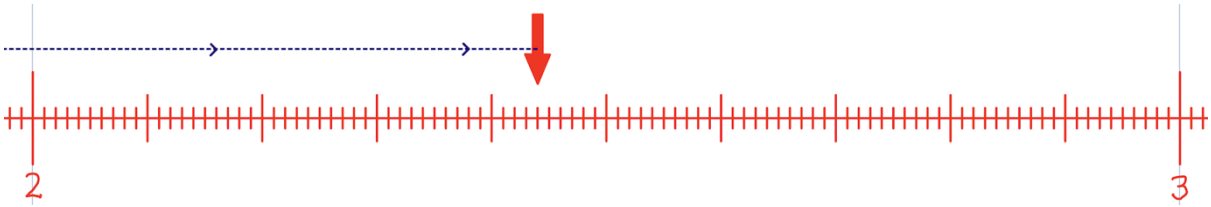
Again, it is difficult to be accurate, so the best we can say is the red arrow has now moved

$$\textit{more than } 2 + \frac{8}{10} \text{ pch and } \textit{less than } 2 + \frac{9}{10} \text{ pch.}$$

How could we be more accurate with our measurements in the last two cases? 🤔

We could sub-divide the $\frac{1}{10}$s into smaller divisions; but how many? Ten.

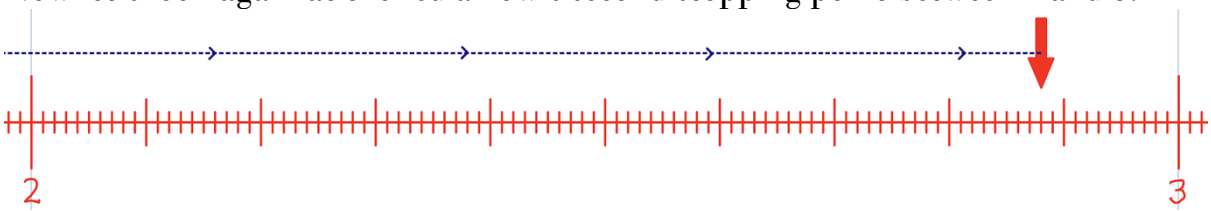
So, each 10 sub-divisions of 1 pch will have 10 smaller (equal) sub-divisions.
Thus, each of the smaller divisions will be $\frac{1}{100}$ pch.



So now, looking at the red arrow's first stopping point in between 2 and 3, we can now see it had travelled *exactly*

$$2 + \frac{4}{10} + \frac{4}{100} \text{ pch}$$

Now let's look again at the red arrow's second stopping point between 2 and 3.



We can now see it has travelled *exactly*

$$2 + \frac{8}{10} + \frac{8}{100} \text{ pch}$$

What if the red arrow stopped a teeny-tiny bit to the right of where it is immediately above? Just past the $\frac{8}{100}$ mark and on the $\frac{9}{100}$ mark? Mmmm. ☺

So, the travelsome red arrow started at zero and made stops after:

$$\frac{3}{10}\text{ pch, } 1+\frac{6}{10}\text{ pch, } 2+\frac{4}{10}+\frac{4}{100}\text{ pch and } 2+\frac{8}{10}+\frac{8}{100}\text{ pch}$$

You might be wondering a couple of things, like:

- Why did we not simplify those fractions?
- Why did we write and + sign and not just write $1\frac{6}{10}$ or $\frac{16}{10}$ or $\frac{8}{5}$?

We did that so you could see (or be reminded of) how the **decimal system** extends to non-whole numbers.

$$1+\frac{6}{10}\text{ pch can be written as 1.6 pch}$$

$$2+\frac{4}{10}+\frac{4}{100}\text{ pch} = 2.44\text{ pch}$$

$$\frac{3}{10}\text{ pch} = 0.3\text{ pch}$$

Every fraction can be written in decimal form.

If the faction has a denominator of 10, or 100, or 1000 or … then it is straightforward to write it as a decimal.

For example:

$$\frac{7}{10} = 0.7$$

$$\frac{72}{100} = \frac{70}{100}+\frac{2}{100} = \frac{7}{10}+\frac{2}{100} = 0.72$$

$$\frac{743}{100} = \frac{700}{100}+\frac{40}{100}+\frac{3}{100} = 7+\frac{4}{10}+\frac{3}{100} = 7.43$$

$$\frac{51}{1000} = \frac{50}{1000}+\frac{1}{1000} = \frac{5}{100}+\frac{1}{1000} = 0.051$$

Question 1

Write each of the following fractions in decimal form.

a) $\frac{2}{10}$ b) $\frac{25}{100}$ c) $4\frac{231}{1000}$ d) $2\frac{4}{100}$ e) $31\frac{21}{1000}$ f) $11\frac{1}{10000}$

14. Fractions as decimals

14.1 Fractions with 'easy' denominators"

Earlier we said every fraction can be written in decimal form. But what if the denominator is not 10, or 100, or 1000 or …?
Such fractions are often called *decimal fractions*.

For example, $\frac{3}{4}$?

Well, we could change $\frac{3}{4}$ to an equivalent fraction with denominator 100 by multiplying both the numerator and denominator by 25.

So,

$$\frac{3}{4} = \frac{75}{100} = \frac{70}{100} + \frac{5}{100} = \frac{7}{10} + \frac{5}{100} = 0.75$$

Another example,

$$\frac{1}{20} = \frac{5}{100} = 0.05$$

Question 1
Write each of the following fractions in decimal form.

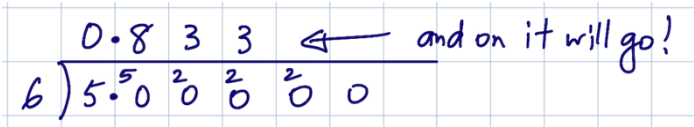
a) $\frac{1}{4}$ b) $\frac{17}{20}$ c) $\frac{31}{50}$ d) $5\frac{43}{50}$ e) $34\frac{2}{25}$ f) $42\frac{16}{25}$

14.2 Fractions with 'not-so-easy' denominators"

Consider the fraction $\frac{5}{6}$.
6 does not easy multiply up to 10 or 100 or …, so what to do?

Recall that we can think about $\frac{5}{6}$ as $5 \div 6$.

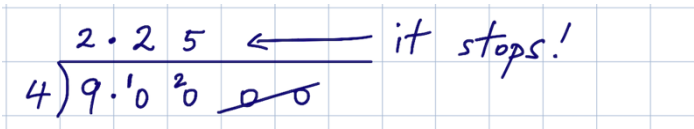
We can use this idea to convert any fraction to a decimal, in this case by dividing 5 by 6. We can do that using either the short or long division algorithm:



So, $\frac{5}{6} = 0.8\overline{3}$

The bar above the 3, indicates that there will be an endless string of 3s after the first 3. We say, *zero point eight three recurring*.
It is called a *vinculum*!

Another example, to convert $\frac{9}{4}$ to a decimal, even though it has an "easy denominator":

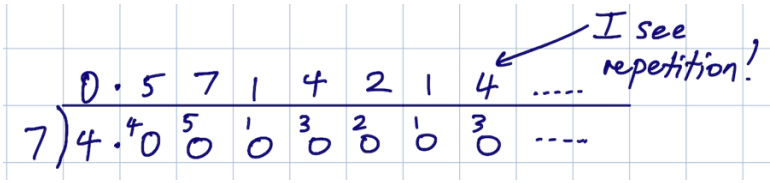


So, $\frac{9}{4} = 2.25$

Now you might be wondering why that works like that, yes?
If you would like to understand more about why that works, please ask your teacher to show you the

"cake" video. ☺

One last example; what is $\frac{4}{7}$ in decimal form?



So, $\frac{4}{7} = 0.57\overline{142}$

Question 1

Use a division algorithm to convert each of the following fractions to decimal form.

a) $\frac{5}{8}$ c) $\frac{1}{9}$ e) $\frac{2}{7}$ g) $\frac{1}{32}$

b) $\frac{5}{6}$ d) $\frac{1}{7}$ f) $\frac{1}{28}$ h) $\frac{1}{81}$

Answers

To come.